Brownian motion and Itô calculus

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Introduction to filtering 2010

Brownian motion and Itô calculus

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- Brownian paths and their properties
    1. Physical Brownian motion
    2. Weak Brownian motion
    3. Strong Brownian motion
    4. Pathwise properties
    5. Functions of bounded variation

5 November

- The Itô calculus

1-Physical Brownian motion

Model of molecular bombardment :
Free motion with independent normal impulses ($t_i \leq t < t_{i+1}$)

$$X_t = X_{t_i} + \delta w_i (t - t_i) \qquad \delta w_i \sim \mathcal{N}(0, (t_{i+1} - t_i)^{-1})$$

Random walk : Sum of independent normal steps

$$X_{t_{i+1}} = X_{t_i} + w_i \qquad w_i \sim \mathcal{N}(0, t_{i+1} - t_i)$$

L^2 **limit** : v_i *i.i.d* and $\sim \mathcal{N}(0,1)$

$$e_i = \frac{\mathbf{1}_{]t_i, t_{i+1}]}}{(t_{i+1} - t_i)^{\frac{1}{2}}} \qquad X_t = \sum_i v_i \langle \mathbf{1}_{]0,t]}, e_i \rangle$$

2-Weak Brownian motion

Definition :

- $X_0 = 0$
- $X_t - X_s \sim \mathcal{N}(0, t-s)$ for $s \leq t$
- $X_t - X_s$ independent of $\{X_u, u \leq s\}$

Associated martingales : Z characterises X

$$Y_t = X_t^2 - t \qquad Z_t = \exp\left(i(u, X_t) + \frac{t}{2}u^2\right)$$

Isonormal process : (e_i ONB in $L^2(0,1)$)

$$X_t = \sum_i v_i \langle \mathbf{1}_{]0,t]}, e_i \rangle \qquad v_i \sim \mathcal{N}(0,1)$$

Wiener integral :

$$L^2(0,1) \ni f = \sum_i e_i \langle f, e_i \rangle \mapsto \sum_i v_i \langle f, e_i \rangle \in L^2(\Omega)$$

3-Strong Brownian motion

Lévy-Ciesielski construction (Haar ONB)

$$e_{n,k}(\tau) = 2^{-\frac{n-1}{2}} \mathbf{1}_{](k-1)2^{-n},k2^{-n}]} - 2^{-\frac{n-1}{2}} \mathbf{1}_{]k2^{-n},(k+1)2^{-n}]}$$

Wiener construction (Trigonometric ONB)

$$e_n(\tau) = \sqrt{2} \cos(2n\pi\tau)$$

Kolmogorov-Chentsov theorem (1/2-Hölder continuity)

$$\mathbb{E}|X_t - X_s|^a \leq C_k(t-s)^{1+b}$$

Holds for all $a = 2k$ and $b = k - 1$

4-Pathwise properties

Stability : If X is Brownian motion then so are the following

$$X_t^{\alpha} = \alpha^{\frac{-1}{2}} X_{\alpha t} \quad X_s^{\tau} = X_t - X_{t \wedge \tau} \quad Y_t = tX(t^{-1})$$

Small/large time : $X \leftrightarrow Y$

$$\tau_+ = \inf\{t, X_t > 0\} = 0 \quad \tau_- = \inf\{t, X_t < 0\} = 0 \ \ (a.s.)$$

Non monotone : Probability that X is increasing on $[0, 1]$

$$\mathbb{P}\left(\bigcap_n \bigcap_{k \leq n} \left\{ X_{\frac{k}{n}} - X_{\frac{(k-1)}{n}} > 0 \right\} \right) = \lim_n 2^{-n} = 0$$

5-Functions of bounded variation

Definition : $|f|_t < \infty$

$$|f|_t = \sup_{\pi} \sum_{t_i \in \pi} |f_{t_i} - f_{t_{i-1}}| \quad \pi \text{ partition of } [0,t]$$

Characterisation : f is difference of two increasing functions

$$f_t = |f|_t - (|f|_t - f_t)$$

Riemann-Stieltjes itnegral : (finite variation gives bound)

$$\sup_{\pi} \left| \sum_{t_i \in \pi} g(t_i)[f(t_{i+1} - t - i)] \right| \leq \|g\|_\infty |f|_t$$

Brownian paths are not of finite variation ! ! ! ! !