Discrete time martingales

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Introduction to filtering 2010

Discrete time martingales

8 October

- Introduction to discrete time martingales

15 October

- Detection theory
  1. Change of probability
  2. Positive martingales
  3. Convergence
- Strong law of large numbers

1-Change of probability

Two probability measures : $(\Omega, \mathcal{A})$ given

$$\mathbb{P}^0 : \mathcal{A} \to [0,1] \qquad \mathbb{P}^1 : \mathcal{A} \to [0,1]$$

Decision uses observations : $\mathcal{F}_n = \sigma(X_m; m \leq n)$
Likelihood Ratios :

$$\mathbb{P}^i_n = \mathbb{P}^i | \mathcal{F}_n \qquad L_n = \frac{d\mathbb{P}^1_n}{d\mathbb{P}^0_n}$$

Probability density : Assumed to exist !

$$\mathbb{E}_1[Y] = \mathbb{E}_0[L_n Y] \qquad Y = f(X_1, \ldots, X_n)$$

Example : $X \hookrightarrow \mathcal{N}(0,1) / X \hookrightarrow \mathcal{N}(m,1)$

$$L = \exp[mX - (m^2/2)] \qquad \mathbb{E}[L] = 1$$

2-Positive martingales

L is a martingale w.r.t $\mathcal{F}$ (and $\mathbb{P}_0$) :
Adaptedness and integrability are clear. We prove,

$$\mathbb{E}_0[L_{n+1}|\mathcal{F}_n] = L_n$$

- $\mathbb{E}_0[L_{n+1}f(X_1,\ldots,X_n)] = \mathbb{E}_1[f(X_1,\ldots,X_n)]$
- $Y = f(X_1,\ldots,X_n)$ is $\mathcal{F}_n$-measurable

$$\mathbb{E}_1[f(X_1,\ldots,X_n)] = \mathbb{E}_0[L_n f(X_1,\ldots,X_n)]$$

More generally :
An adapted process X is a $\mathbb{P}_1$-martingale iff LX is a $\mathbb{P}_0$-martingale

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- $Y = f(X_1, \ldots, X_n)$ is $\mathcal{F}_n$-measurable

$$\mathbb{E}_1[f(X_1, \ldots, X_n)] = \mathbb{E}_0[L_n f(X_1, \ldots, X_n)]$$

Example : $X = (X_1, \ldots, X_n)$ and "independent"

$$L_n = \exp[(m, X) - (|m|^2/2)]$$

3-Convergence

Is $\mathbb{P}^1$ absolutely continuous w.r.t. $\mathbb{P}^0$?

$$L_\infty \text{ exists } \Leftrightarrow L_n = \mathbb{E}[L_\infty | \mathcal{F}_n]$$

A more general question :

$$\text{Given } L \text{ positive martingale, } \mathbb{E}[L_n] = 1, \text{ find } \mathbb{P}^1$$

Log-likelihood : Under condition of integrability

$$l_n = -\log(L_n) \text{ is a submartingale w.r.t. } \mathbb{P}^0$$

Example : (continued)

$$l_n = -\sum_{k=1}^{n} m_k X_k + \frac{1}{2} \sum_{k=1}^{n} m_k^2$$

3-Convergence

Is $\mathbb{P}^1$ absolutely continuous w.r.t. $\mathbb{P}^0$?

$$L_\infty \text{ exists } \Leftrightarrow L_n = \mathbb{E}[L_\infty | \mathcal{F}_n]$$

A more general question :

Given L positive martingale, $\mathbb{E}[L_n] = 1$, find $\mathbb{P}^1$

Log-likelihood : Under condition of integrability

$$l_n = -\log(L_n) \text{ is a submartingale w.r.t. } \mathbb{P}^0$$

If "independent" : by law of large numbers

$$\frac{1}{n} l_n \text{ provides perfect test}$$

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  1. Borel-Cantelli lemma
  2. 0-1 law
  3. Lévy's downward theorem

1-Borel-Cantelli lemma

Sequence of events : $A_n \in \mathcal{A}, n \geq 1$

$$\overline{A} = \cap_{n\geq 1} \cup_{m\geq n} A_n \qquad \underline{A} = \cup_{n\geq 1} \cap_{m\geq n} A_m$$

Borel-Cantelli :

$$\sum_{n\geq 1} \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(\overline{A}) = 0$$

If A_n independent :

$$\mathbb{P}(\overline{A}) = 0 \Rightarrow \sum_{n\geq 1} \mathbb{P}(A_n) < \infty$$

2- 0-1 law

Tail σ-algebra : $\mathcal{T}_n = \sigma(X_m, m \geq n)$

$$X_n \textit{ i.i.d.} \Rightarrow \cap_{n\geq 1}\mathcal{T}_n \text{ trivial}$$

Conclusion : $S_n = (1/n)\sum_{m\leq n} X_m$

- X_n converge with probability $0, 1$
- (Eventual) limit is a constant
- Same holds for S_n

Tentative proof :

- Weak law of large numbers : $S_n \xrightarrow{\mathbb{P}} \mathbb{E}[X_1]$
- If a.s. limit exists, it must be the same
- Failure !

3-Lévy's downward theorem

A direct result of martingale convergence

Filtration :

$$\mathcal{F}_{-n} = \sigma(S_m, m \geq n)$$

Martingale convergence : (a.s.)

$$\mathbb{E}[X_1|\mathcal{F}_{-n}] \to \mathbb{E}[X_1|\mathcal{F}_{-\infty}]$$

Independence : (0-1 law)

$$\mathbb{E}[X_1|\mathcal{F}_{-\infty}] = \mathbb{E}[X_1]$$

Conclusion : (stationarity)

$$(1/n)\sum_{k=1}^{n} X_k = \mathbb{E}[X_1|\mathcal{F}_{-n}]$$