Discrete time martingales

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Introduction to filtering 2010

Discrete time martingales

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- Optimal detection
- Law of large numbers

1-History and importance

The three structures of probability

- Stationary
- Markov
- Martingale

History

- Study of sums of i.i.d random variables (1930s)
- The word "martingale" due to Ville (1930s-1940s)
- General theory by Doob (1951)
- Continuous time martingales by Meyer (1975)

Importance

- Engineering
- Finance
- Applied maths

2-Basic concepts

A martingale is a process respecting a time flow of information

Filtration : Probability space $(\Omega, \mathcal{A}, \mathbb{P})$, Time set $T \subset \mathbb{R}$, a filtration

$$(\mathcal{F}_t)_{t \in T} \qquad \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{A} \text{ for } s \leq t \in T$$

Adapted process : $(X_t)_{t \in T}$ with $\sigma(X_t) \subset \mathcal{F}_t$

Natural filtration :

$$\mathcal{F}_t^X = \sigma(X_s; s \leq t)$$

Martingale : (X w.r.t $\mathcal{F}$)

- X is adapted to $\mathcal{F}$
- X is integrable $\mathbb{E}|X_t| < \infty$
- Martingale property $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$
- Submartingales, Supermartingales

3-Examples

i.i.d. sum : $(U_n)_{n\geq 0}$

$$X_n = U_0 + \ldots + U_n \text{ according to sign of } \mathbb{E}[U_i]$$

i.i.d. product :

$$X_n = U_0 \ldots U_n \qquad \mathbb{E}[U_i] = 1$$

Closed martingale :

$$\xi \text{ integrable } \mathcal{F} \text{ a filtration } X_t = \mathbb{E}[\xi|\mathcal{F}_t]$$

Martingale transform : (discrete time)

$$(C.X)_n = \sum_{k\leq n} C_k(X_k - X_{k-1})$$

C is a previsible process : “stake on previous game”

4-Centering

As of now $T = \mathbb{N}$*. A martingale is a process with no trends*
Let $X = (X_n)_{n \geq 0}$ be a process and $\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$ a filtration.
Assume X is adapted and integrable.

$$A_n = \sum_{k=1}^{n} \mathbb{E}[X_k - X_{k-1} | \mathcal{F}_{k-1}] \qquad A_0 = 0$$

Define a new process $\tilde{X} = (\tilde{X}_n)_{n \geq 0}$

$$\tilde{X}_n = X_n - A_n$$

- X is a submartingale *iff* A is increasing.
- A is *previsible*, adapted to $(\mathcal{F}_{n-1})_{n \geq 0}$ $-\mathcal{F}_0 = \{\phi, \Omega\}$.

More advanced results :

- Continuous time, the Doob-Meyer decomposition
- Removal of drift by change of measure

5-Convex maps

Stability of submartingales :

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$$

Binary operations :

- Martingales are stable by linear combination
- Submartingales are stable by positive combination
- Maximum of two submartingales is a submartingale

Convex maps : $f(X)$ is a submartingale if integrable and

- X a martingale and f convex
- X a submartingale and f convex increasing

Example : X^2 where X is a submartingale.

$$X_n^2 - [X]_n \text{ where } [X] \text{ quadratic variation}$$

6-Optional times

What are the events that can be detected from observation $\mathcal{F}$
Random time : $\tau : \Omega \to T$ **Optional time** : $\mathbf{1}_{[\tau,\infty[}$ is an adapted process

$$\text{For } n \geq 0 \text{ we have } \{\tau \leq n\} \in \mathcal{F}_n$$

Examples : $(A \in \mathbb{R})$

- First time that $|X_t| \geq A$ (optional)
- Last time that $|X_t| \geq A$ (not optional)

Corresponding σ-field : $A \in \mathcal{F}_\tau$

$$\text{For } n \geq 0 \text{ we have } A \cap \{\tau \leq n\} \in \mathcal{F}_n$$

Optional evaluation : $X_\tau(\omega)$ is $\mathcal{F}_\tau$-measurable

7-Optional stopping

Martingale property is conserved by random time change

Let τ_1, τ_2 be optional times.

$$\tau_1 \leq \tau_2 \Rightarrow \mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$$

Let X be a martingale. Assume τ_2 bounded. Then,

$$X_{\tau_i} \text{ is integrable and } \mathbb{E}[X_{\tau_2}|\mathcal{F}_{\tau_1}] = X_{\tau_1}$$

Proof : Let $u \geq \tau_2 \geq \tau_1$

$$\mathbb{E}[X_u|\mathcal{F}_{\tau_2}] = X_{\tau_2} \qquad \mathbb{E}[X_u|\mathcal{F}_{\tau_1}] = X_{\tau_1}$$

Conclusion : By projection property.

Generalisation : What if τ_2 not bounded ?

8-Inequalities

Maximum inequality :

X a martingale, X_n^* running maximum

$$r\mathbb{P}\left(X_n^* \geq r\right) \leq \mathbb{E}\left[X_n; X_n^* \geq r\right] \leq \mathbb{E}|X_n|$$

Hitting probability : $\tau = \inf\{n, X_n \geq r\}$

$$\{X_n^* \geq r\} = \{\tau \leq n\} \in \mathcal{F}_{\tau \wedge n}$$

Optional stopping :

$$\mathbb{E}[X_{\tau \wedge n}; \tau \leq n] \leq \mathbb{E}[X_n; \tau \leq n]$$

Chebychev inequality :

$$r\mathbb{P}\left(X_n^* \geq r\right) \leq r\mathbb{P}\left(\tau \leq n\right) \leq \mathbb{E}[X_{\tau \wedge n}; \tau \leq n]$$

8-Inequalities

Doob inequality : X a martingale

$$\mathbb{E}\left(|X|_n^*\right)^2 \leq 4\mathbb{E}|X_n|^2$$

Upcrossing inequality :
$N(a, b; n)$ =number of crossings upwards of $[a, b]$

$$\mathbb{E}N(a, b; n) \leq \mathbb{E}(X_n - a)^+/b - a$$

Inequalities and convergence : X converges *iff*

- It is bounded
- It does not “oscillate” infinitely often

Convergence

Almost sure convergence :
If $\sup_n \mathbb{E}|X_n| < \infty$ then X converges almost surely

$$\lim_n X_n(\omega) = X_\infty(\omega)$$

Convergence in L^1 :
X converges in L^1 *iff* X is closed

$$X_n = \mathbb{E}[X_\infty | \mathcal{F}_n]$$

Upward theorem :

$$X_n = \mathbb{E}[\xi | \mathcal{F}_n] \Rightarrow X_\infty = \mathbb{E}[\xi | \mathcal{F}_\infty]$$